

PROJECTIVELY FULL IDEALS IN NOETHERIAN RINGS, A SURVEY

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Abstract

We discuss projective equivalence of ideals in Noetherian rings and the existence or failure of existence of projectively full ideals. We describe connections with the Rees valuations and Rees integers of an ideal, and consider the question of whether improvements can be made by passing to an integral extension ring.

1 Definitions, summary and examples.

Let R be a Noetherian ring and let $I \neq R$ be a regular ideal of R . (So I contains an element with zero annihilator.)

Definition 1.1 An ideal J of R is **projectively equivalent** to I if there exist positive integers m and n such that I^m and J^n have the same integral closure, i.e., $(I^m)_a = (J^n)_a$.

Projective equivalence is an equivalence relation on the regular proper ideals of R . Let $\mathbf{P}(I)$ denote the set of integrally closed ideals projectively equivalent to I .

Remark 1.2 The set $\mathbf{P}(I)$ is discrete and linearly ordered with respect to inclusion. Moreover, J and K in $\mathbf{P}(I)$ and m and n positive integers implies $(J^m K^n)_a \in \mathbf{P}(I)$. Thus there is naturally associated to I and $\mathbf{P}(I)$ a unique subsemigroup $S(I)$ of the additive semigroup of nonnegative integers \mathbb{N}_0 such that $S(I)$ contains all sufficiently large integers. A semigroup having these properties is called a **numerical semigroup**. It is observed in [3, Remark 3.11] that every numerical semigroup S is realizable as $S(M)$, where (R, M) is a local domain of altitude one.

Definition 1.3 The set $\mathbf{P}(I)$ is said to be **projectively full** if $S(I) = \mathbb{N}_0$, or equivalently, if every element of $\mathbf{P}(I)$ is the integral closure of a power of the largest element K of $\mathbf{P}(I)$, i.e., every element of $\mathbf{P}(I)$ has the form $(K^n)_a$, for some positive integer n . If this holds, then each ideal J in R such that $J_a = K$ is said to be **projectively full**.

Our main goals may be listed as follows:

1. Describe ideals K that are projectively full.
2. Describe ideals I such that $\mathbf{P}(I)$ is projectively full.
3. Determine if improvements can be obtained by passing to an integral extension ring.

In connection with the third main goal, the following theorem is proved in [4].

Theorem 1.4 *If R contains the field of rational numbers, then there exists a finite free integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full; and if R is an integral domain, then there also exists a finite integral extension domain B of R such that $\mathbf{P}(IB)$ is projectively full.*

A strengthened version of Theorem 1.4 is given in Corollary 7.5 below.

Examples 1.5 and 1.6 illustrate aspects of the projectively full property.

Example 1.5 Let (R, M) be a Noetherian local ring having the property

$$a \in M^i \setminus M^{i+1} \text{ and } b \in M^j \setminus M^{j+1} \implies ab \notin M^{i+j+1},$$

then M is projectively full. Thus if the associated graded ring

$$\mathbf{G}(R, M) = R/M \oplus M/M^2 \oplus \cdots \oplus M^n/M^{n+1} \oplus \cdots$$

is an integral domain, then M is projectively full.

Example 1.6 Let x be an indeterminate over the field F and let $R = F[[x^2, x^3]]$. Then the maximal ideal $M = (x^2, x^3)R$ is not projectively full. The numerical semigroup $S(M)$ is generated by 2 and 3.

Notice that with $R = F[[x^2, x^3]]$ as in Example 1.6, the ring R is not integrally closed. Things improve by passing to the integral extension ring $R[x] = F[[x]]$. Indeed, for each regular proper ideal I of R , $\mathbf{P}(IR[x])$ is projectively full.

Example 1.7 is established in [3] to demonstrate existence of a normal local domain (R, M) such that M is not projectively full.

Example 1.7 Let F be a field in which 2 and 3 are units and let x, y, z, w be variables. Let

$$R_0 = F[x, y]_{(x, y)} \quad \text{and} \quad R = \frac{R_0[z, w]}{(z^2 - x^3 - y^3, w^2 - x^3 + y^3)}.$$

Then R is a normal local domain of altitude¹ two with maximal ideal $M = (x, y, z, w)R$, and M is not projectively full.

2 Some history.

The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in [17] and further developed by Nagata in [14]. Rather than ‘projectively equivalent’, Hartmut Göhner uses the term ‘asymptotically equivalent’ in [6]. Göhner mentions that the expression ‘projective asymptotic equivalence’ is used by David Rees in [16] and by H. T. Muhly in [13].

Let I be a regular proper ideal of a Noetherian ring R . For each $x \in R$, let $v_I(x) = \max\{k \in \mathbb{N} \mid x \in I^k\}$, and define

$$\bar{v}_I(x) = \lim_{k \rightarrow \infty} \left(\frac{v_I(x^k)}{k} \right).$$

Rees established that:

- (a) $\bar{v}_I(x)$ is well defined.
- (b) For each $k \in \mathbb{N}$ and $x \in R$, $\bar{v}_I(x) \geq k$ if and only if $x \in (I^k)_a$.
- (c) There exist discrete valuations v_1, \dots, v_n defined on R , (with values in $\mathbb{N} \cup \infty$) and positive integers e_1, \dots, e_n such that, for each $x \in R$, $\bar{v}_I(x) = \min\{\frac{v_i(x)}{e_i} \mid i = 1, \dots, n\}$.

¹We are following Nagata [15] and using altitude for what is often now termed dimension or Krull dimension.

In (c), we say that v is a valuation on R if $\{x \in R \mid v(x) = \infty\}$ is a prime ideal P , $v(x) = v(y)$ if $x + P = y + P$, and the function v induces on R/P is a valuation.

For simplicity, assume R is a Noetherian integral domain with field of fractions F . Let t be an indeterminate. The **Rees ring** of R with respect to I is the graded subring

$$\mathbf{R} = R[t^{-1}, It] = \bigoplus_{n \in \mathbb{Z}} I^n t^n$$

of the Laurent polynomial ring $R[t^{-1}, t]$. The integral closure B of $R[t^{-1}, It]$ is a Krull domain, and B_P is a DVR for each minimal prime P of $t^{-1}B$. The set **Rees I of Rees valuation rings** of I is precisely the set of rings $V = B_P \cap F$, where P is a minimal prime of $t^{-1}B$.

Definition 2.1 Let $(V_1, N_1), \dots, (V_n, N_n)$ be the Rees valuation rings of I . The integers (e_1, \dots, e_n) , where $IV_i = N_i^{e_i}$, are the **Rees integers** of I . (These are the same e_i as in (c) above.)

A sufficient condition for I to be projectively full is that $\gcd(e_1, \dots, e_n) = 1$. However, this is not a necessary condition as we demonstrate in Example 3.1.

3 Projective equivalence and Rees valuation rings.

Example 3.1 Let (R, M) be a regular local ring of altitude two with $M = (x, y)R$, and let $e > 1$ be an integer. The ideal $I = (x, y^e)R$ is integrally closed with unique Rees valuation ring $V = R[x/y^e]_{MR[x/y^e]}$. The integrally closed ideals projectively equivalent to I are precisely the powers I^n of I . Thus I is projectively full. The maximal ideal of V is $N = yV$ and $IV = N^e$, so the gcd of the Rees integers of I is $e > 1$.

If I and J are projectively equivalent, then $\text{Rees } I = \text{Rees } J$. The converse holds if I or J has only one Rees valuation ring, but fails in general. Steve McAdam, Jack Ratliff and Judy Sally show in [11] that if I and J are projectively equivalent, then the Rees integers of I and J are proportional. It is observed in [2] that the converse also holds: if $\text{Rees } I = \text{Rees } J$ and the Rees integers of I and J are proportional, then I and J are projectively equivalent.

Example 3.2 Assume (R, M) is a regular local ring of altitude two. Zariski's theory of unique factorization of complete (= integrally closed) ideals of R as finite products of simple complete ideals implies $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I of R . The ideal I has a unique Rees valuation ring if and only if I is a power of a simple complete ideal. If I factors as $I = I_1^{f_1} \cdots I_n^{f_n}$, where I_1, \dots, I_n are distinct simple complete ideals, then I is projectively full if and only if $\gcd(f_1, \dots, f_n) = 1$.

How do the integers f_1, \dots, f_n relate to the Rees integers of I ? The simple complete M -primary ideals of R are in one-to-one correspondence with the prime divisors birationally dominating R . Thus the Rees valuation rings of I are $(V_1, N_1), \dots, (V_n, N_n)$, where (V_j, N_j) corresponds to I_j . If $I_j V_j = N_j^{c_j}$, then the Rees integers of I are $e_1 = c_1 f_1, \dots, e_n = c_n f_n$.

Example 3.3 illustrates these concepts in a specific situation.

Example 3.3 Assume (R, M) is a regular local ring of altitude two with $M = (x, y)R$ as in Example 3.1. Let $I = (x^2, xy^2, y^3)R$. Notice that $J = (x^2, y^3)R$ is a reduction of I and $JI = I^2$, so the reduction number $r_J(I) = 1$. Let

$$V = R\left[\frac{xy^2}{x^2}, \frac{y^3}{x^2}\right]_{MR\left[\frac{xy^2}{x^2}, \frac{y^3}{x^2}\right]} = R\left[\frac{y^2}{x}, \frac{y^3}{x^2}\right]_{MR\left[\frac{y^2}{x}, \frac{y^3}{x^2}\right]}.$$

One sees that V is a valuation ring with maximal ideal $N = (y^2/x)V$, and I is a simple complete ideal. The ideals of R that are contracted from V descend as follows: $M = N^2 \cap R \supsetneq (x, y^2)R = N^3 \cap R \supsetneq M^2 = N^4 \cap R \supsetneq (x, y^2)M = N^5 \cap R \supsetneq I = N^6 \cap R$. The ideals in $\mathbf{P}(I)$ are precisely the ideals $I^m = N^{6m} \cap R$, for $m \in \mathbb{N}$.

4 The Rees integers and the associated graded ring.

The following result is established in [4, Example 5.1].

Proposition 4.1 *If $\mathbf{G}(R, I) = R/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots$ has a minimal prime z such that z is its own z -primary component of (0) , then I has a Rees integer equal to one. Therefore I is projectively full. In particular, if $\mathbf{G}(R, I)$ is reduced, then I is projectively full.*

More can be said about the Rees integers of I by using the Rees ring $\mathbf{R} = R[t^{-1}, It]$ and the identification $\mathbf{G}(R, I) = \mathbf{R}/t^{-1}\mathbf{R}$. Let \mathbf{R}' denote the integral closure of \mathbf{R} . Proposition 4.2 is from [4, Proposition 4.1].

Proposition 4.2 *The ideal I has a Rees integer equal to one if and only if $t^{-1}\mathbf{R}$ has a minimal prime p such that $t^{-1}\mathbf{R}'_p = p\mathbf{R}'_p$.*

Let (R, M) be a normal local domain and let $\mathbf{G}(R, M)$ denote the associated graded ring of R with respect to M . Proposition 4.1 implies that M is projectively full if $\mathbf{G}(R, M)$ is reduced. Example 4.3 shows that the converse of this statement is not true.

Example 4.3 An example of a normal local domain (R, M) of altitude two such that M is projectively full and the associated graded ring $\mathbf{G}(R, M)$ is not reduced. Let F be an algebraically closed field with $\text{char } F = 0$, and let R_0 be a regular local domain of altitude two with maximal ideal $M_0 = (x, y)R_0$ and coefficient field F , e.g., $R_0 = F[x, y]_{(x, y)}$, or $R_0 = F[[x, y]]$. Then $V_0 = R_0[y/x]_{xR_0[y/x]}$ is the unique Rees valuation ring of M_0 . Let $R = R_0[z]$, where $z^2 = x^3 + y^j$ with $j \geq 3$. It is readily checked that R is a normal local domain of altitude two with maximal ideal $M = (x, y, z)R$. Moreover, the image of z in M/M^2 is a nonzero nilpotent element of the associated graded ring $\mathbf{G}(R, M)$, so $\mathbf{G}(R, M)$ is not reduced.

Notice that $I = (x, y)R$ is a reduction of M since z is integral over I . It follows that every Rees valuation ring of M is an extension of V_0 . Let V be a Rees valuation ring of M and let v denote the normalized valuation with value group \mathbb{Z} corresponding to V . Then $v(x) = v(y)$ and the image of y/x in the residue field of V is transcendental over F . Since $z^2 = x^3 + y^j$ and $j \geq 3$, we have

$$2v(z) = v(z^2) = v(x^3 + y^j) = 3v(x).$$

We conclude that $v(x) = 2$ and $v(z) = 3$. Therefore V is ramified over V_0 . This implies that V is the unique extension of V_0 in the quadratic field extension generated by z over the field of fractions of R_0 , and thus the unique Rees valuation ring of M .

For each positive integer n , let $I_n = \{r \in R \mid v(r) \geq n\}$. Thus $I_2 = M$. Since V is the unique Rees valuation ring of M , we have $I_{2n} = (M^n)_a$ for each $n \in \mathbb{N}$.

To show M is projectively full, we prove that V is not the unique Rees valuation ring of I_{2n+1} for each $n \in \mathbb{N}$. Consider the inclusions

$$M^2 \subseteq I_4 \subset (z, x^2, xy, y^2)R := J \subseteq I_3 \subset I_2 = M.$$

Since $\lambda(M/M^2) = 3$ and since the images of x and y in M/M^2 are F -linearly independent, $J = I_3$ and $M^2 = I_4 = (M^2)_a$.

Since $x^3 = z^2 - y^j$ and $j \geq 3$, $L = (z, y^2)R$ is a reduction of $I_3 = (z, x^2, xy, y^2)R$. Indeed, $(x^2)^3 \in L^3$ and $(xy)^3 \in L^3$ implies x^2 and xy are integral over L . It follows that V is not a Rees valuation of I_3 , for $zV \neq y^2V$.

Consider $M^3 \subset I_3M \subseteq I_5 \subset I_4 = M^2$. Since the images of x^2, xy, y^2, xz, yz in M^2/M^3 are an F -basis, it follows that $I_3M = I_5$ and $M^3 = (M^3)_a = I_6$. Proceeding by induction, we assume $M^{n+1} = (M^{n+1})_a = I_{2n+2}$, and that I_{2n+1} is not projectively equivalent to M . Consider

$$M^{n+2} \subset I_3M^n \subseteq I_{2n+3} \subset I_{2n+2} = M^{n+1}.$$

Since the images in M^{n+1}/M^{n+2} of $\{x^a y^b \mid a + b = n + 1\} \cup \{zx^a y^b \mid a + b = n\}$ is an F -basis, $\lambda(M^{n+1}/M^{n+2}) = 2n + 3$, and the inequalities $\lambda(M^{n+1}/I_{2n+3}) \geq n + 2$ and $\lambda(I_3M^n/M^{n+2}) \geq n + 1$ imply $I_3M^n = I_{2n+3}$ and $M^{2n+2} = (M^{2n+2})_a$.

Therefore the ideal I_{2n+3} has a Rees valuation ring different from V , and thus is not projectively equivalent to M . We conclude that M is projectively full. We have also shown that M is a normal ideal.

5 Questions and examples in altitude two.

In Questions 5.1 we list several questions that interest us and that we hope may stimulate further research work in this area.

Questions 5.1 Let (R, M) be a complete normal local domain of altitude two.

1. What are necessary and sufficient conditions in order that M be projectively full?
2. Let I be an M -primary ideal. What are necessary and sufficient conditions in order that I be projectively full?

3. What are necessary and sufficient conditions in order that $\mathbf{P}(I)$ be projectively full for each M -primary ideal I of R ?
4. What are necessary and sufficient conditions in order that $\mathbf{P}(I)$ be projectively full for each nonzero proper ideal I of R ?

In Example 5.2, we present in more detail a family of examples from [3, Example 5.1] that includes Example 1.7.

Example 5.2 An example of a (complete) normal local domain (R, M) of altitude two such that M is not projectively full. Let F be an algebraically closed field, and let R_0 be a regular local domain of altitude two with maximal ideal $M_0 = (x, y)R_0$ and coefficient field F , e.g., $R_0 = F[x, y]_{(x, y)}$, or $R_0 = F[[x, y]]$. Let $k < i$ be relatively prime positive integers ≥ 2 , and let $R = R_0[z, w]$, where $z^k = x^i + y^i$ and $w^k = x^i - y^i$. Assume that 2, i and k are units of F . It is readily checked that R is a normal local domain of altitude two with maximal ideal $M = (x, y, z, w)R$. Also R is a free R_0 -module of rank k^2 .

With $R = R_0[z, w]$ as above, we show that $M = (x, y, z, w)R$ has a unique Rees valuation ring and that M is not projectively full. Notice that $L = (x, y)R$ is a reduction of M , for $z^k \in (x^i, y^i)R \subseteq L^k$ and $w^k \in (x^i, y^i)R \subseteq L^k$ imply z and w are integral over L . Thus each Rees valuation ring V of M is an extension of the order valuation ring $V_0 = R_0[y/x]_{xR_0[y/x]}$ of R_0 . To show there exists a unique Rees valuation ring of M , we observe that if V is a Rees valuation ring of M , then V as an extension of V_0 ramifies of degree k and undergoes a residue field extension of degree at least k .

To show V ramifies of degree k over V_0 , observe that $kv(z) = v(z^k) = v(x^i + y^i) = iv(x) = iv(y)$ implies $v(z) = i$ and $v(x) = v(y) = k$. Similarly, $v(w) = i$. Let N denote the maximal ideal of V . We have $(z, w)V = N^i$ and $MV = (x, y)V = N^k$. The residue field of V_0 is $F(\bar{\tau})$, where $\bar{\tau}$ is the image of $\tau = y/x$ and is transcendental over F . Now w/z is a unit of V and

$$\left(\frac{w}{z}\right)^k = \frac{x^i - y^i}{x^i + y^i} = \frac{1 - \tau^i}{1 + \tau^i}.$$

It follows that the residue class of w/z in V/N is algebraic of degree k over $F(\bar{\tau})$. This proves that V is the unique Rees valuation ring of M .

To show M is not projectively full, notice that $(z^k, w^k)R = (x^i + y^i, x^i - y^i)R$, and since $\text{char } F \neq 2$, $(x^i + y^i, x^i - y^i)R = (x^i, y^i)R$. Since $(x^i, y^i)R$ is a reduction of M^i , we have $((z^k, w^k)R)_a = (M^i)_a$. Also $((z^k, w^k)R)_a = ((z, w)^k R)_a$. Therefore $(z, w)R$ and M are projectively equivalent, so V is the unique Rees valuation ring of $(z, w)R$, and $((z, w)R)_a = N^i \cap R$. We have $M = N^k \cap R$, $M^n V = N^{nk}$ and $(M^n)_a = N^{nk} \cap R$, for each positive integer n . Since i is not a multiple of k , M is not projectively full.

Remark 5.3 Let (R, M) be a normal local domain of altitude two. Muhly and Sakuma in [12] define **the filtration** $\{A_i \mid i \in \mathbb{N}_0\}$ **associated to a prime R -divisor v** by $A_0 = R$ and $A_i = \{x \in R \mid v(x) > v(A_{i-1})\}$ for $i \geq 1$. The prime R -divisor v is said to be **Noetherian** if the associated filtration $\{A_i \mid i \geq 0\}$ possesses a Noetherian subfiltration, that is a subfiltration $\{A_{i_j} \mid j \geq 0\}$ such that the graded ring $\bigoplus_{j \geq 0} A_{i_j}/A_{i_{j+1}}$ is Noetherian. Muhly and Sakuma say that a normal local domain R **satisfies condition N** if each prime divisor of the second kind relative to R is Noetherian [12, First sentence of Section 4]. They show that if R is analytically normal of altitude two, then each prime divisor of the second kind relative to R is the unique valuation ring of some ideal I of R if and only if R satisfies condition N [12, Proposition 2.2 and the paragraph following it]. (It is shown in [1] that some caution should be used in reading the given proof of their needed result [12, Proposition 2.1].) Muhly and Sakuma also show in [12] that if an analytically normal local domain R of altitude two satisfies condition (N), then there is a factorization theory for complete M -primary ideals that extends some aspects of Zariski's factorization theory for complete ideals in the case where R is regular. Muhly carries this work further in [13] by showing that the hypothesis that R is analytically normal can be weakened to R is normal, and also that contrary to what was conjectured in [12], R being analytically normal does not imply R satisfies condition (N).

Assume (R, M) is a complete normal local domain of altitude two. Göhner in [6] proves that R satisfies condition (N) if R is semifactorial, i.e., if $\text{Pic}R$ is a torsion group, and conjectures that the converse also holds. Cutkosky proves this conjecture in [5]. This establishes an interesting connection between the divisorial ideals of R and the complete M -primary ideals.

Göhner [6, Corollary 4.5] observes that if (R, M) is a complete normal local domain of altitude two and if R/M is the algebraic closure of a finite field, then R is semifactorial and therefore satisfies condition (N). By taking F in Example 5.2 to be the algebraic closure of a finite field in which 2, i and k are units, we see that there exists a complete normal local domain (R, M) that satisfies condition (N) and has the property that its maximal ideal M is not projectively full.

Remark 5.4 Let (R, M) be a normal local domain of altitude two and let I be an M -primary ideal. Recall that the multiplicity $e(I) = e(I, R)$ of I is a positive integer [10, Section 14]. In connection with Questions 5.1, Joe Lipman pointed out to us that if $e(I)$ is squarefree, then I is projectively full. For an ideal and its integral closure have the same multiplicity, and for each positive integer n , $e(I^n) = n^2e(I)$ [10, Formula 14.3]. If m and n are relatively prime positive integers and J is an ideal such that $(I^m)_a = (J^n)_a$, then $m^2e(I) = n^2e(J)$, and the assumption that $e(I)$ is squarefree implies that $n = 1$. Therefore the hypothesis that $e(I)$ is squarefree implies that I is projectively full.

More generally, let (R, M) be a local domain of altitude $d \geq 2$ and let I be an M -primary ideal. For each positive integer n , we have $e(I^n) = n^de(I)$. Assume that m and n are relatively prime positive integers and that J is an ideal such that $(I^m)_a = (J^n)_a$. Then $m^de(I) = n^de(J)$. If $e(I)$ is free of d -th powers, then $n = 1$. Therefore the hypothesis that $e(I)$ is free of d -th powers implies that I is projectively full.

Notice that for (R, M) as in Example 5.2, we have $e(M) = k^2$. In particular, for each integer $k \geq 2$, Example 5.2 establishes the existence of a normal local domain (R, M) of altitude two such that M is not projectively full and $e(M) = k^2$.

6 Rational singularities and the closed fiber.

Lipman in his paper [9] extends Zariski's theory of complete ideals in a regular local domain of altitude two to the case where R is a normal local domain of altitude two that has a rational singularity. Lipman proves that R has unique factorization of complete ideals if and only if the completion of R is a UFD. For R having this property, it follows that $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I , e.g. this is the case if $R = F[[x, y, z]]$,

where $z^2 + y^3 + x^5 = 0$ and F is a field of characteristic zero.

Let (R, M) be a normal local domain of altitude two. Göhner proves that if R has a rational singularity, then the set of complete asymptotically irreducible ideals associated to a prime R -divisor v consists of the powers of an ideal A_v which is uniquely determined by v . In our terminology, this says that if I is a nonzero proper ideal of R having only one Rees valuation ring, then $\mathbf{P}(I)$ is projectively full. Göhner's proof involves choosing a desingularization $f : X \rightarrow \operatorname{Spec} R$ such that v is centered on a component E_1 of the closed fiber on X .

Let E_2, \dots, E_n be the other components of the closed fiber on X . Let E_X denote the group of divisors having the form $\sum_{i=1}^n n_i E_i$, with $n_i \in \mathbb{Z}$, and consider

$$E_X^+ = \{D \in E_X \mid D \neq 0 \text{ and } (D \cdot E_i) \leq 0 \text{ for all } 1 \leq i \leq n\}, \text{ and}$$

$$E_X^\# = \{D \in E_X \mid D \neq 0 \text{ and } O(-D) \text{ is generated by its sections over } X\}.$$

Lipman shows $E_X^\# \subseteq E_X^+$ and equality holds if R has a rational singularity. Also, if $D = \sum_i n_i E_i \in E_X^+$, then negative-definiteness of the intersection matrix $(E_i \cdot E_j)$ implies $n_i \geq 0$ for all i .

For if $D \in E_X^+$ and $D = A - B$, where A and B are effective, then $(A - B \cdot B) \leq 0$ and $(A \cdot B) \geq 0$ imply $(B \cdot B) \geq 0$, so $B = 0$.

Let $v = v_1, v_2, \dots, v_n$ denote the discrete valuations corresponding to E_1, \dots, E_n . Associated with $D = \sum_i n_i E_i \in E_X^\#$ one defines the complete M -primary ideal

$$I_D = \{r \in R \mid v_i(r) \geq n_i \text{ for } 1 \leq i \leq n\}.$$

This sets up a one-to-one correspondence between elements of $E_X^\#$ and complete M -primary ideals that generate invertible O_X -ideals.

Lipman suggested to us the following proof that $\mathbf{P}(I)$ is projectively full for each complete M -primary ideal I if R has a rational singularity. Fix a desingularization

$$f : X \rightarrow \operatorname{Spec} R$$

such that I generates an invertible O_X -ideal and let $D = \sum_i n_i E_i \in E_X^\#$ be the divisor associated to I . Let $g = \gcd\{n_i\}$. Since $E^+ = E^\#$, it follows that $(1/g)D \in E^\#$.

The ideals $J \in \mathbf{P}(I)$ correspond to divisors in $E^\#$ that are integral multiples of $(1/g)D$. Thus if K is the complete M -primary ideal associated to $(1/g)D$, then each $J \in \mathbf{P}(I)$ is the integral closure of a power of K , so $\mathbf{P}(I)$ is projectively full.

7 Integral extensions.

Perhaps the main unresolved question related to projectively full ideals is the following:

Question 7.1 Let I be a nonzero ideal of a Noetherian domain R . Does there always exist a finite integral extension domain B of R such that $\mathbf{P}(IB)$ is projectively full?

An affirmative answer to Question 7.1 is obtained in [4, Theorem 2.5] if the ideal I has a Rees integer that is a unit of R . More precisely, let $I = (b_1, \dots, b_g)R$ be a nonzero regular ideal of the Noetherian ring R , let $R_g = R[X_1, \dots, X_g]$ and $K = (X_1^c - b_1, \dots, X_g^c - b_g)R_g$, where c is a positive integer. Then $A = R_g/K$ is a finite free integral extension ring of rank c^g of R . Let $x_i = X_i \bmod K$. Then $J = (x_1, \dots, x_g)A$ is such that $(IA)_a = (J^c)_a$, so IA and J are projectively equivalent.

Theorem 7.2 *Let $I = (b_1, \dots, b_g)R$ be as above and let (V, N) be a Rees valuation ring of I . Assume that $b_i V = IV = N^c$ for each i with $1 \leq i \leq g$, and that c is a unit of R . Then the finite free integral extension ring $A = R_g/K$ is such that $J = (x_1, \dots, x_g)A$ is projectively full. Hence $\mathbf{P}(IA)$ is projectively full. If R is an integral domain and z is a minimal prime of A , then $B = A/z$ is an integral extension domain of R such that $\mathbf{P}(IB)$ is projectively full.*

Let I be a regular proper ideal in a Noetherian ring R and let e be the least common multiple of the Rees integers of I . Consider the Rees ring $R[t^{-1}, It]$ and let B denote the integral closure of $R[t^{-1}, It][t^{-\frac{1}{e}}]$ in $R[t^{-\frac{1}{e}}, t^{\frac{1}{e}}]$. Shiroh Itoh in [8, page 392] proves that $t^{-\frac{1}{e}}B$ is a radical ideal, and $B/t^{-\frac{1}{e}}B$ is a reduced graded ring.

This result of Itoh motivated us to ask about the existence of an integral extension ring A of R in which there exists a radical ideal that is projectively equivalent to IA . The following two results are established in [7].

Theorem 7.3 *Let I be a regular proper ideal in a Noetherian ring R and let b_1, \dots, b_g be regular elements in R that generate I . Let m be a positive integer and let*

$$K = (X_1^m - b_1, \dots, X_g^m - b_g)R[X_1, \dots, X_g].$$

For $i = 1, \dots, g$ let x_i be the residue class of X_i modulo K , let $A = R[x_1, \dots, x_g]$, and let $J = (x_1, \dots, x_g)A$. Then:

(7.3.1) *A is a finite free integral extension ring of R and $(J^m)_a = (IA)_a$, so J is projectively equivalent to IA .*

(7.3.2) *$(J^k)_a \cap R = I_{\frac{k}{m}}$ ($= \{x \in R \mid \bar{v}_I(x) \geq \frac{k}{m}\}$) for all positive integers k .*

(7.3.3) *If e is the least common multiple of the Rees integers of I and $m = e$, then J_a is a radical ideal.*

(7.3.4) *If R is an integral domain and if z is a minimal prime ideal in A , then $(J + z)/z$ is projectively equivalent to $(IA + z)/z$, and if $m = e$ as in (7.3.3), then $(J_a + z)/z$ is a radical ideal.*

Theorem 7.4 *Let I be a regular proper ideal in a Noetherian ring R , let V_1, \dots, V_n be the Rees valuation rings of I , and let e_1, \dots, e_n be the corresponding Rees integers of I . Assume that the least common multiple e of e_1, \dots, e_n is a unit in R and that there exists a regular element b in I such that $bV_i = IV_i$ for $i = 1, \dots, n$.*

(7.4.1) *Let b_1, \dots, b_g be regular elements in R that generate I , let $A = R[x_1, \dots, x_g]$ ($= R[X_1, \dots, X_g]/(X_1^e - b_1, \dots, X_g^e - b_g)$), and let $J = (x_1, \dots, x_g)A$. Then A is a finite free integral extension ring of R , $J_a = \text{Rad}(IA)$ is a projectively full radical ideal that is projectively equivalent to IA , $(IA)_a = (J^e)_a$, and all Rees integers of J are equal to one.*

(7.4.2) *If R is a Noetherian domain, then let $b_1 = b$, b_2, \dots, b_g be a basis of I , for $i = 1, \dots, g$ fix an e -th root $b_i^{1/e}$ of b_i in an algebraic closure of the quotient field of R , and let $A = R[b_1^{1/e}, \dots, b_g^{1/e}]$ and $J = (b_1^{1/e}, \dots, b_g^{1/e})A$. Then A is a Noetherian domain that is a finite integral extension of R , $\text{Rad}(IA) = J_a$ is a projectively full radical ideal that is projectively equivalent to IA , $(IA)_a = (J^e)_a$, and all Rees integers of J are equal to one.*

We have the following corollary to Theorem 7.4

Corollary 7.5 *Let I be a regular nonzero ideal in a Noetherian ring R and assume that R contains the field of rational numbers. Then there exists a finite free integral extension ring A of R that contains a projectively full radical ideal J that is projectively equivalent to IA and whose Rees integers are all equal to one. If R is an integral domain, then for each minimal prime ideal z in A the ideal $(J+z)/z$ in A/z is a projectively full radical ideal that is projectively equivalent to $(IA+z)/z$, and all Rees integers of $(J+z)/z$ are equal to one.*

The following example indicates difficulties in establishing a result similar to Theorem 7.4 without the hypothesis that e is a unit of R .

Example 7.6 Let $D = k[y]$, where k is a field of characteristic two and y is an indeterminate, and let $b = y^2(y-1)$. Then D is a PID, and $I = bD$ has the two Rees integers $e_1 = 1$ (for its Rees valuation ring D_{p_1} , where $p_1 = (y-1)D$) and $e_2 = 2$ (for its Rees valuation ring D_{p_2} , where $p_2 = yD$), so the least common multiple of e_1, e_2 is $e = 2$. Let E be the integral closure of $D[x]$ in its quotient field, where $x = \sqrt{b}$. Then E is a Dedekind domain and the Rees integer of xE with respect to E_{yE} is 2, so xE is not a radical ideal of E .

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